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FORMULATION OF BOUNDARY-VALUE PROBLEMS OF STATICS FOR THIN ELASTIC ASYMMETRICALLY-LAMINATED ANISOTROPIC PLATES AND SOLUTION USING FUNCTIONS OF A COMPLEX VARIABLE†

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The internal stress–strain state (SSS) of a plate under coupled bending and extension–compression-shear is determined. It is proved that the two-dimensional equilibrium equations for the materials of layers with anisotropy of general form are elliptic, and natural boundary conditions are derived. The displacements are expressed in terms of functions of a complex variable, which make it possible to reduce the basic boundary-value problems to a determination of these functions from the values of the real part of linear combinations of the functions on the lateral surface of the plate. Conditions are established for the functions to be single-valued; these conditions are related to the self-balance of the boundary load. Exact solutions of the boundary-value problems are presented for a finite ellipse, including the case where polynomial loads are applied to the plate faces. Solutions are presented for a plate with an elliptic cut-out, including the case in which the boundary loads are not self-balanced. The singular problem of a plate with a loaded finite cut through the material or a rigid insertion is also considered. Exact solutions are constructed, using Schwartz's formula, for simply-connected regions that can be mapped conformally onto a disk. The method proposed is a new variation of the Kolosov–Muskhelishvili–Lekhnitskii method for non-classical laminated plates and has the same advantages and disadvantages. The main difference compared with the standard two-dimensional problem or bending problem is the higher dimension.

1. Consider a plate consisting of N perfectly bonded layers whose materials possess rectilinear anisotropy of a general type. Suppose that the jth layer occupies a region $\Omega[z_j, z_{j+1}]$, $\mathbf{x} \in \Omega \subseteq R^2$ (where $\mathbf{x} = \mathbf{i}_\alpha x_\alpha$, $x_3 \equiv z$ are Cartesian coordinates). Denote the stiffness matrix and layer thickness by $\mathbf{G}_j$ and $h_j = z_{j+1} - z_j$, respectively. We shall assume that the ratio of the plate half-thickness h to the characteristic longitudinal dimension L of the strain pattern is a small parameter ε, and that the ratios of the geometric and elastic constants of the layers are incommensurate with ε. The normal and tangential loads on the lateral surfaces of the plate are determined by equalities

$$\sigma_{zz} = \sigma^{\mp}(\mathbf{x}), \quad \sigma_{\alpha z} = \varepsilon^{-1}\tau_\alpha^{\mp}(\mathbf{x}); \quad z_{1,N+1} = z^{\mp} \tag{1.1}$$

In [1–3] we established asymptotically exact ($\varepsilon \to 0$) two-dimensional equations describing the SSS of a plate in which the layers are asymmetrically stacked by thickness. The strains and stresses were expanded in asymptotic series in powers of ε; the principal terms for the longitudinal displacements $\mathbf{U}$ and deflection W turned out to be the same in all the layers. The resolvent equations of statics are

$$\begin{aligned} &\partial_\beta \chi_{\alpha\beta}(\mathbf{D}_1)\mathbf{u} - \partial_\beta \chi_{\alpha\beta}(\mathbf{D}_2)\,\mathrm{grad}\, W + T_\alpha = 0 \\ &-\partial^2_{\alpha\beta}\chi_{\alpha\beta}(\mathbf{D}_2)\mathbf{u} + \partial^2_{\alpha\beta}\chi_{\alpha\beta}(\mathbf{D}_3)\,\mathrm{grad}\, W = T \end{aligned} \tag{1.2}$$

$$\begin{aligned} &T_\alpha = \varepsilon^{-1}(\tau_\alpha^+ - \tau_\alpha^-), \quad T = \sigma^+ - \sigma^- + \varepsilon^{-1}\,\mathrm{div}(z^+\boldsymbol{\tau}^+ - z^-\boldsymbol{\tau}^-) \\ &\mathbf{U} = \mathbf{u}(\mathbf{x}) - z\,\mathrm{grad}\, W, \quad W = w(\mathbf{x}) \\ &e_{\alpha\beta} = \varepsilon_{\alpha\beta} + z\theta_{\alpha\beta}, \quad \varepsilon_{\alpha\beta} = \tfrac{1}{2}(\partial_\beta u_\alpha + \partial_\alpha u_\beta), \quad \theta_{\alpha\beta} = -\partial^2_{\alpha\beta} w \\ &\sigma^j_{\alpha\beta} = \chi_{\alpha\beta}(\boldsymbol{\Gamma}_j)(\mathbf{u} - z\,\mathrm{grad}\, w), \quad (Q_{\alpha\beta}, M_{\alpha\beta}) = \sum_j \int_{z_{j+1}}^{z_j} (1, z)\sigma^j_{\alpha\beta}dz \\ &Q_{\alpha z} = \sum_j \int_{z_j}^{z_{j+1}} \sigma^j_{\alpha z}dz = \partial_\beta M_{\alpha\beta} + \varepsilon^{-1}(z^+\tau_\alpha^+ - z^-\tau_\alpha^-); \quad \alpha, \beta = 1, 2 \end{aligned} \tag{1.3}$$

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$$\begin{Vmatrix} Q_{\alpha\beta} \\ M_{\alpha\beta} \end{Vmatrix} = \begin{Vmatrix} \mathbf{D}_1 & \mathbf{D}_2 \\ \mathbf{D}_2 & \mathbf{D}_3 \end{Vmatrix} \begin{Vmatrix} \nu_{\alpha\beta} \\ \kappa_{\alpha\beta} \end{Vmatrix}, \quad \begin{matrix} \nu_{\alpha\beta} = \varepsilon_{\alpha\beta}(1+\delta^{\beta}_{\alpha+1}) \\ \kappa_{\alpha\beta} = \theta_{\alpha\beta}(1+\delta^{\beta}_{\alpha+1}) \\ \alpha\beta = 11, 12, 22 \end{matrix} \tag{1.4}$$

$$\chi_{11}(\Gamma) = \mathbf{i}_1(\gamma_{11}\partial_1 + \gamma_{16}\partial_2) + \mathbf{i}_2(\gamma_{16}\partial_1 + \gamma_{12}\partial_2) \quad (1 \leftrightarrow 2)$$

$$\chi_{12}(\Gamma) = \mathbf{i}_1(\gamma_{16}\partial_1 + \gamma_{66}\partial_2) + \mathbf{i}_2(\gamma_{66}\partial_1 + \gamma_{26}\partial_2)$$

$$\mathbf{D}_k = \frac{1}{k}\sum_j (z^k_{j+1} - z^k_j)\Gamma_j, \quad \mathbf{D} = \begin{Vmatrix} \mathbf{D}_1 & \mathbf{D}_2 \\ \mathbf{D}_2 & \mathbf{D}_3 \end{Vmatrix}$$

$$\Gamma = \begin{Vmatrix} \gamma_{11} & \gamma_{16} & \gamma_{12} \\ \gamma_{16} & \gamma_{66} & \gamma_{62} \\ \gamma_{12} & \gamma_{62} & \gamma_{22} \end{Vmatrix}, \quad \gamma_{pq} = \pm\frac{G^p_q}{G_0}, \quad G_0 = G\begin{pmatrix} 3 & 4 & 5 \\ 3 & 4 & 5 \end{pmatrix} \tag{1.5}$$

where $\sigma^j_{\alpha\beta}$, $e_{\alpha\beta}$ are the stresses and strains, $Q_{\alpha\beta}$, $M_{\alpha\beta}$ are the stress resultants and moments in the plate cross-section ($\delta^{\beta}_{\alpha+1}$ is the Kronecker delta) and $Q_{\alpha z}$ is the shear stress resultant. The quantities γ^j_{pq} define the mean stiffnesses in the jth layer, G_0 is the principal minor of the stiffness matrix **G** and G^p_q is the minor obtained by adding the pth row and qth column to G_0 in the natural order (choosing the minus sign when pq = 16.26). The cumbersome expressions for the stresses $\sigma^i_{\alpha\beta}$, σ^j_{zz} are not written out.

The asymptotic error of formulae (1.2)–(1.4) comprises $O(\varepsilon)$ for the case of general anisotropy and $O(\varepsilon^2)$ for layers with local longitudinal planes of symmetry of the elastic properties.

Note that the error estimate is independent of the stacking and holds even for a single-ply plate.

2. The main difference compared with classical Kirchhoff–Love plate theory is that, besides membrane stiffnesses ($\mathbf{D}_1$) and bending stiffnesses ($\mathbf{D}_3$), there are also membrane-bending stiffnesses $\mathbf{D}_2 \neq 0$. The processes of bending and extension–compression-shear turn out to be coupled. The minimization of this coupling and the optimal placing of the reference system with respect to thickness were studied in [1, 2].

We shall prove that, as far as the basic properties are concerned, continuity is maintained relative to the classical case. Some of them are obvious from physical considerations, but deserve special formulation.

Theorem 1. The generalized stiffness matrices Γ_j of the layers are positive-definite.

That the principal minors of the matrix Γ_j are positive follows from the following considerations:

1. All the principal minors of the initial matrix **G** are positive.
2. The quantities $\pm G^p_q$ are equal to the values of the bordered minors, obtained by adding the pth row and qth column to G_0 below and to the right.
3. The bordered minors satisfy Sylvester's identity [4]: if $M = |a_{ij}|^m_1, n > m \geqslant 1, n \geqslant 2$, is a principal minor of the matrix $A = |a_{ij}|^n_1$ and b_{kl} is the bordered minor obtained by adding to M the kth row and lth column, then the determinant of the matrix of bordered minors $B = |b_{kl}|^m_{p+1}$ satisfies the equality $B = M^{n-p-1}A$.

Theorem 2. The energy density in each layer and the specific elastic energy Π of the plate as a whole are positive-definite

$$\pi_j = \tfrac{1}{2}\sigma^j_{\alpha\beta}e_{\alpha\beta} \geqslant 0, \quad \Pi = \tfrac{1}{2}\int_\Omega Q_{\alpha\beta}\varepsilon_{\alpha\beta} + M_{\alpha\beta}\theta_{\alpha\beta}d\Omega \tag{2.1}$$

This follows from Theorem 1, the equalities $(\sigma_{11}, \sigma_{12}, \sigma_{22})^m_j = \Gamma_j(\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{22})^m$ = and (1.4).

As a corollary, we obtain the following.

Theorem 3. The complete stiffness matrix **D**, bending stiffness matrix $\mathbf{D}_3$ and membrane stiffness matrix $\mathbf{D}_1$ are positive-definite.

Theorem 4. The energy balance relation holds for the statics of the plate

$$\Pi = \int_\Omega T_\alpha u_\alpha + Twd\Omega + \int_{\partial\Omega} Q_n u_n + Q_\tau u_\tau + M_n\theta_n + P_n w d\tau \tag{2.2}$$

$$P_n = Q_{nz} + \partial_\tau M_\tau, \quad \theta_n = -\partial_n w$$

where **n** and τ are the vectors of the outward normal and tangent to the contour $\partial\Omega$.

To prove this, multiply Eqs (1.2) by the displacements u_α and w and integrate over the volume $\Omega\,[z, z^+]$, applying the Stokes and Gauss formulae.

Theorem 5. Suppose that the contour $\partial\Omega$ is divided into segments $\partial\Omega = \cup_k L_k$, on each of which a condition of following type holds

$$(1)\ u_n^*, u_\tau^*, \theta_n^*, w^*; \quad (2)\ Q_n^*, Q_\tau^*, M_n^*, P_n^*; \quad (3) \text{ mixed combinations} \tag{2.3}$$

Then the SSS of the plate is uniquely defined.

Indeed, the difference between any two solutions will have energy $\Pi = 0$, so that $\varepsilon_{\alpha\beta} = \theta_{\alpha\beta} = 0$.

Asymptotic analysis of the boundary conditions and the interaction of the boundary layer with the internal SSS for laminated plates is an extremely complicated problem. Nevertheless, it is well known from the example of single-ply plates [5–7] that the natural Kirchhoff boundary conditions hold to within an error $O(\varepsilon)$. In laminated plates a modification of the boundary conditions is required if the following situation arises [8]:

1. the distribution of the boundary loads with respect to thickness is strongly non-uniform;
2. the properties of the layers are considerably different, to the extent that new small (large) parameters commensurate with $\varepsilon^{\pm 1}$ may appear.

In the case of "smooth" distributions and comparatively homogeneous properties of the layers, one can confine one's attention to conditions (2.3).

3. We will now present the solution of a boundary-value problem for a clamped plate in the shape of a finite ellipse

$$\partial\Omega: f(\mathbf{x}) \equiv (x_1 / a_1)^2 + (x_2 / a_2)^2 - 1 = 0$$

$$u_n^* = u_\tau^* = w^* = 0, \quad \theta_n^* = 0$$

when the given loads on the lateral surfaces are represented by homogeneous polynomials $T_\alpha^m(\mathbf{x})$, $T^{m-1}(\mathbf{x})$. Set $u_\alpha = \upsilon_\alpha^m(\mathbf{x})f(\mathbf{x})$, $w = \upsilon^{m-1}(\mathbf{x})f^2(\mathbf{x})$, where υ_α^m, υ^{m-1} are also homogeneous polynomials in x_1 and x_2 of degrees m and $m-1$, respectively. The displacements satisfy the boundary conditions identically and contain $2(m+1) + m = 3m + 2$ undetermined constants—the coefficients of the powers of x_α in the polynomials. Substituting into Eqs (1.2) and collecting like terms we obtain a linear system of equations of order $3m + 2$ for the coefficients. That the system is non-singular follows from Theorem 5. In particular, putting $T_\alpha = \text{const}$, $T = 0$, we get $w = 0$, $u_1 = ((a_{11}T_1 - a_{12}T_2)/(a_{11}a_{22} - a_{12}^2))f(\mathbf{x})$ $(1 \leftrightarrow 2)$ $\mathbf{i}_\beta a_{\alpha\beta} \equiv \partial_\beta \chi_{\alpha\beta}(D_1)(f, f)^m$.

This result also yields a number of particular solutions when the lateral load is the same but the boundary conditions on the contour of the ellipse are different.

4. We will now propose a general method for solving boundary-value problems (2.3), on the assumption that there are no front loads.

Theorem 6. The symbols of the differential operator of system (1.2), as well as its "bending" and "membrane" components, are elliptic.

The main idea of the proof is as follows: take the Fourier transformation of Eqs (1.2)

$$(u_\alpha, w) = \frac{1}{\sqrt{2\pi}} \iint (V_\alpha, V) e^{i\mathbf{s}\mathbf{x}} dx_1 dx_2, \quad \mathbf{s} = \mathbf{i}_\alpha s^\alpha$$

$$\mathbf{L}(\partial_1, \partial_2)(u_1, u_2, w) = 0 \Rightarrow \mathbf{L}(is^1, is^2)(V_1, V_2, V) = 0$$

Apart from signs and the replacement of V_α by $\pm iV_\alpha$, the symbolic equation will be equivalent to the equality $\mathbf{L}(s^1, s^2)(V_1, V_2, V)e^{\mathbf{s}\mathbf{x}} = 0$. The SSS of the plate determined by displacements $(u_\alpha, w) = (c_\alpha, c)e^{\mathbf{s}\mathbf{x}}$ has energy

$$\Pi = (c_1, c_2, c) \int_\Omega \mathbf{L}(s^1, s^2) e^{2\mathbf{s}\mathbf{x}} d\Omega (c_1, c_2, c)^m \geqslant 0$$

with equality at zero displacements only. Consequently, $\mathbf{L}(s^1, s^2) \neq 0$, $\mathbf{L}(is^1, is^2) \neq 0$ for $s^\alpha \in R$. As a special case we conclude that the bending and membrane components of the operator are elliptic.

It follows that the eigenvalues of the operator as a whole, and also of its "classical" components, must be complex numbers.

The general solution will be sought in the form $u_\alpha = u_\alpha(\mathbf{sx})$, $w = w(\mathbf{sx})$. To fix our ideas, let us assume that $s^1 = 1$, $s^2 = \lambda$, retaining both sets of notation for convenience. It follows from Eqs (1.2) that

$$\begin{Vmatrix} p_{11} & p_{12} & -p_{13} \\ p_{12} & p_{22} & -p_{23} \\ -p_{13} & -p_{23} & p_{33} \end{Vmatrix} \begin{Vmatrix} u_1^{\mathrm{III}} \\ u_2^{\mathrm{III}} \\ w^{\mathrm{IV}} \end{Vmatrix} = 0, \quad \begin{matrix} p_1 = p_{13}p_{22} - p_{12}p_{23} \\ p_0 = p_{11}p_{22} - p_{12}^2 \quad (1 \leftrightarrow 2) \end{matrix}$$

$$\mathrm{i}_\beta p_{\alpha\beta} = e^{-\mathbf{sx}} \partial_\beta \chi_{\alpha\beta}(\mathbf{D}_1) e^{\mathbf{sx}} \tag{4.1}$$

$$p_{\alpha 3} = e^{-\mathbf{sx}} \partial_\beta \chi_{\alpha\beta}(\mathbf{D}_2)\, \mathrm{grad}\, e^{\mathbf{sx}}$$

$$p_{33} = e^{-\mathbf{sx}} \partial^2_{\alpha\beta} \chi_{\alpha\beta}(\mathbf{D}_3)\, \mathrm{grad}\, e^{\mathbf{sx}}$$

$$p = p_0 p_{33} - p_{22} p_{13}^2 - p_{11} p_{23}^2 + 2 p_{12} p_{23} p_{31}$$

The fourth-degree characteristic polynomials p_{33} and p_0 correspond to the bending and membrane operators, respectively; the eight-degree polynomial p corresponds to the total operator. The roots of the equation $p = 0$ fall into four conjugate pairs; we shall assume throughout that $\mathbf{s}_k \neq \mathbf{s}_m$. The structure of the displacements, stresses and moments is as follows (summation over the subscripts $k = 1, 2, 3, 4$ only)

$$u_\alpha = 2\,\mathrm{Re}\left\{ \Sigma \frac{p_\alpha}{p_0} (\mathbf{s}_k) \psi_k'(\zeta_k) \right\} + \upsilon_\alpha^0(\mathbf{x}), \quad w = 2\,\mathrm{Re}\{\Sigma \psi_k(\zeta_k)\} \tag{4.2}$$

$$p(\mathbf{s}_k) = 0, \quad \zeta_k = \mathbf{s}_k \mathbf{x}, \quad \mathrm{Im}\, \lambda_k > 0$$

$$(Q_{\alpha\beta}, M_{\alpha\beta}) = 2\,\mathrm{Re}\left\{ \Sigma (q_{\alpha\beta}, m_{\alpha\beta})(\mathbf{s}_k) \psi_k''(\zeta_k) \right\} + (Q_{\alpha\beta}^0, M_{\alpha\beta}^0)$$

$$Q_{\alpha 3} = 2\,\mathrm{Re}\left\{ \Sigma q_{\alpha 3}(\mathbf{s}_k) \psi_k'''(\zeta_k) \right\}; \quad Q_{\alpha\beta}^0, M_{\alpha\beta}^0(\upsilon_1, \upsilon_2) = \mathrm{const} \tag{4.3}$$

$$\frac{s^1}{s^2} m_{11} + 2 m_{12} + \frac{s^2}{s^1} m_{22} = 0, \quad \frac{q_{11}}{(s^2)^2} = -\frac{q_{12}}{s^1 s^2} = \frac{q_{22}}{(s^1)^2} \tag{4.4}$$

$$\frac{q_{13}}{s^2} = -\frac{q_{23}}{s^1}, \quad q_\alpha = s^\beta m_{\alpha\beta} \tag{4.5}$$

where $q_{\alpha\beta}(\mathbf{s}_k)$ and $m_{\alpha\beta}(\mathbf{s}_k)$ are rational functions of the eigenvalues of the operator (1.2); $\upsilon_\alpha^0(\mathbf{x})$ are linear functions of x_1 and x_2.

We shall now show how to set up the main boundary-value problems (2.3).

First boundary-value problem. Suppose we are given the displacements on the contour $\partial\Omega$. Then the functions $\psi'_k(\zeta_k)$ satisfy the equations

$$2\,\mathrm{Re}\left\{ \Sigma \frac{p_\alpha}{p_0} (\mathbf{s}_k) \psi_k'(\zeta_k) \right\} = u_\alpha^* - \upsilon_\alpha^0(\mathbf{x})$$

$$2\,\mathrm{Re}\left\{ \Sigma s_k^\alpha \psi_k'(\zeta_k) \right\} = -\theta_\alpha^* \tag{4.6}$$

$$u_\alpha = n_\alpha u_n - n_\beta u_\tau \quad (u_\alpha \leftrightarrow \theta_\alpha), \quad n_\alpha = \pm dx_\beta / d\tau \quad (\alpha\beta = 12, 21)$$

Second boundary-value problem. The stresses and moments (2.3) on $\partial\Omega$ are conveniently integrated over an arc $(0^\smile, t) \subset \partial\Omega$ (the initial point is chosen arbitrarily). Using (4.4) and (4.5) and proceeding exactly as in [9, 10] for anisotropic single-ply plates, we obtain the following relations (no summation over repeated indices $\alpha\alpha$)

$$
\begin{gathered}
2\,\mathrm{Re}\left\{\Sigma \frac{q_{\alpha\alpha}}{s_k^\beta}(s_k)\psi_k'(\zeta_k)\right\} = \pm\int\limits_0^t Q_\alpha^* d\tau - Q_{\alpha\alpha}^0 x_\beta + Q_{12}^0 x_\alpha + c_\alpha \\
2\,\mathrm{Re}\left\{\Sigma \frac{m_{\alpha\alpha}}{s_k^\beta}(s_k)\psi_k'(\zeta_k)\right\} = \pm\int\limits_0^t M_\alpha^* d\tau - M_{\alpha\alpha}^0 x_\beta + M_{12}^0 x_\alpha \pm c x_\alpha + c_{\alpha+2} \\
Q_\alpha^* = n_\alpha Q_n^* - n_\beta Q_\tau^*, \quad M_\alpha^* = n_\alpha M_n^* - n_\beta F_n^* \\
F_n^* = \int\limits_0^t p_n^* d\tau; \quad c_1, \ldots, c_4, c = \mathrm{const}
\end{gathered}
\tag{4.7}
$$

Note that the integrals on the right of formulae (4.7) depend only on the initial and final point of the contour, since equalities (4.4) and (4.5) lead to expressions for the total differentials.

5. Equations (4.6) and (4.7) reduce the boundary-value problems to a typical problem of complex analysis: given the real parts of certain expressions on the boundary $\partial\Omega$, it is required to determine the function $\varphi_k'(\zeta_k)$ inside the domain. On changing to the complex variable $\zeta_k = x_1 + \lambda_k x_2$, the functions $\psi_k(\zeta_k)$ may become multivalued—an inadmissible situation for physically meaningful quantities. Letting $\Delta_k, \Delta_k', \ldots$ denote the increments of the functions $\psi_k(\zeta_k), \psi_k'(\zeta_k), \ldots$ at a given point x, with the contour described in the positive sense, we obtain exactly 20 conditions for single-valuedness

$$
w{:}\ 2\mathrm{Re}\{\Sigma\Delta_k\} = 0 \tag{5.1}
$$

$$
\begin{gathered}
M_3{:}\ 2\Sigma\,\mathrm{Re}\left\{\frac{q_{11}}{(s_k^2)^2}(s_k)(\Delta_k - \zeta_k\Delta_k')\right\} + \mathcal{M}_3 = 0 \\
u_\alpha{:}\ 2\,\mathrm{Re}\left\{\Sigma\frac{p_\alpha}{p_0}(s_k)\Delta_k'\right\} = 0, \quad \theta_\alpha{:}\ 2\,\mathrm{Re}\{\Sigma s_k^\alpha \Delta_k'\} = 0 \\
Q_\alpha{:}\ 2\,\mathrm{Re}\left\{\Sigma\frac{q_{\alpha\alpha}}{s_k^\beta}(s_k)\Delta_k'\right\} \pm \mathcal{F}_\alpha = 0 \quad (\alpha = 1, 2) \\
M_\alpha{:}\ 2\Sigma\,\mathrm{Re}\left\{\frac{m_{\alpha\alpha}}{s_k^\beta}(s_k)\Delta_k' - \frac{q_{\alpha 3}}{s_k^\beta}(s_k)x_\alpha\Delta_k''\right\} + \mathcal{M}_\beta = 0 \\
\varepsilon_{\alpha\beta}, \omega_{\alpha\beta}{:}\ 2\,\mathrm{Re}\left\{\Sigma s_k^\alpha \frac{p_\alpha}{p_0}(s_k)\Delta_k''\right\} = 0
\end{gathered}
\tag{5.2}
$$

$$
\begin{gathered}
\theta_{\alpha\beta}{:}\ 2\,\mathrm{Re}\left\{\Sigma s_k^\alpha s_k^\beta \Delta_k''\right\} = 0 \\
\mathcal{F}_3{:}\ 2\,\mathrm{Re}\left\{\Sigma\frac{q_{13}}{s_k^2}(s_k)\Delta_k''\right\} - \mathcal{F}_3 = 0
\end{gathered}
\tag{5.3}
$$

$$
Q_{\alpha 3}{:}\ 2\,\mathrm{Re}\{\Sigma q_{\alpha 3}(s_k)\Delta_k'''\} = 0 \tag{5.4}
$$

where $\mathcal{F}_\alpha$, $\mathcal{F}_3$ and $\mathcal{M}_\alpha$, $\mathcal{M}_3$ denote the projections of the principal vector and principal moment of the boundary load on the coordinate axes; $\omega_{\alpha\beta} = \partial_\beta u_\alpha - \partial_\alpha u_\beta$ denotes the rotation of the plate in the longitudinal plane. Additional single-valuedness conditions, for the stress resultants and moments $Q_{\alpha\beta}$ and $M_{\alpha\beta}$, are not needed, because of equalities (1.4). In a multiply-connected domain, conditions (5.1)–(5.4) are set up for each contour; in addition, the boundary loads on the holes and the external contour must balance one another, and the displacements, angles of rotation, strains and curvatures must be single-valued.

6. For each non-self-balanced load at the contour of a hole, let us choose a certain point $\mathbf{x}_0$: $\zeta_k^0 = \mathbf{s}_k\mathbf{x}_0$ (inside the hole). The multivalued terms are found as

$$\Psi_k(\zeta_k) = (\tfrac{1}{2}A\zeta_k^2 + B\zeta_k + C)\ln(\zeta_k - \zeta_k^0)$$
$$\Delta_k''' = 0, \quad \Delta_k'' = 2\pi iA, \quad \Delta_k' = 2\pi iB + \zeta_k\Delta_k'' \tag{6.1}$$
$$\Delta_k = 2\pi iC + \zeta_k\Delta_k' - \tfrac{1}{2}\zeta_k^2\Delta_k''; \quad A, B, C = \text{const}$$

The constants A and B are determined from Eqs (5.2) and (5.3) and the quantities C are found simultaneously from equalities (5.1) and (4.6) or (4.7).

7. We will now demonstrate the solution of boundary-value problems for a finite ellipse and a plate with an elliptic cutout. We expand the boundary conditions in Fourier series in terms of the angular coordinate φ. The right-hand sides of the equations are (the coefficients with subscripts ± 1 form conjugate pairs)

$$x_1 = a_1\cos\varphi, \quad x_2 = a_2\sin\varphi, \quad n_1 = a_2\rho^{-1}\cos\varphi, \quad n_2 = a_1\rho^{-1}\sin\varphi$$
$$\rho = (a_1^2\sin^2\varphi + a_2^2\cos^2\varphi)^{1/2}, \quad d\tau = \rho d\varphi$$
$$u_\alpha^* = \sum_{-\infty}^{\infty} u_{l,\alpha}e^{il\varphi}, \quad -\theta_\alpha^* = \sum_{-\infty}^{\infty} w_{l,\alpha}e^{il\varphi} \tag{7.1}$$
$$\int_{\partial\Omega} \theta_\alpha^* d\tau = \pi\{a_2(w_{1,2} + w_{-1,2}) + ia_1(w_{1,1} - w_{-1,1})\} \tag{7.2}$$

$$\pm\int_0^t Q_\alpha^* d\tau = \pm\mathcal{F}_\alpha\frac{\varphi}{2\pi} + \sum_{-\infty}^{\infty} q_{l,\alpha}e^{il\varphi} \tag{7.3}$$

$$\mathcal{M}_3 = -\pi\{ia_1(q_{1,2} - q_{-1,2}) - a_2(q_{1,1} + q_{-1,1})\} \tag{7.4}$$

$$\pm\int_0^t M_\alpha^* d\tau = -\frac{\varphi}{2\pi}\left\{\mathcal{M}_\beta \mp a_\alpha\mathcal{F}_3(\delta_\alpha^1\cos\varphi + \delta_\alpha^2\sin\varphi)\right\} + \sum_{-\infty}^{\infty} m_{l,\alpha}e^{il\varphi} \tag{7.5}$$

$$\mathcal{M}_1 = \pi\{a_1(m_1 + m_{-1}) + a_2(f_1 + f_{-1})\} \tag{7.6}$$
$$\mathcal{M}_2 = i\pi\{a_2(m_1 - m_{-1}) + a_1(f_1 + f_{-1})\}$$
$$M_n^* = \sum_{-\infty}^{\infty} m_l e^{il\varphi}, \quad F_n^* = \sum_{-\infty}^{\infty} f_l e^{il\varphi}$$

We shall assume that $\mathcal{M} = 0$, $\mathcal{F} = 0$. The variables ζ_k yield affine transformations of the initial domain Ω, under which the ellipse $\partial\Omega$ changes to a new ellipse $\partial\Omega_k$ ($\Omega \to \Omega_k$).

We will first consider an infinite plate with a cutout. The required functions are expressed in terms of conformal mappings $\eta_k^{-1}(\zeta_k)$ of the exteriors of the ellipses Ω_k onto the unit disk [10–13] (choosing the principal branch of the radical)

$$\eta_k = \frac{\zeta_k + \xi_k}{a_1 - i\lambda_k a_2}, \quad \xi_k = (\zeta_k^2 - e_k^2)^{1/2}, \quad 2\zeta_k = (a_1 - i\lambda_k a_2)\eta_k + (a_1 + i\lambda_k a_2)\eta_k^{-1}$$
$$\partial\Omega\colon \ \eta_k = e^{i\varphi}, \quad \xi_k = i(a_1\sin\varphi + \lambda_k a_2\cos\varphi)$$
$$e_k = (a_1^2 + \lambda_k^2 a_2^2)^{1/2}, \quad b_k = (a_1 + i\lambda_k a_2)(a_1 - i\lambda_k a_2)^{-1}$$
$$\psi_k(\zeta_k) = \text{const} + \psi_{0,k}\zeta_k + \tfrac{1}{2}(a_1 - i\lambda_k a_2)\{\psi_{1,k}\ln\eta_k + \psi_{2,k}\eta_k^{-1} + \sum_2^{\infty}(b_k\psi_{l-1,k} - \psi_{l+1,k})(l\eta_k^l)^{-1}\} \tag{7.7}$$
$$\psi_k' = \sum_0^{\infty}\psi_{l,k}\eta_k^{-l}, \quad \psi_k'|_{\partial\Omega} = \sum_0^{\infty}\psi_{l,k}e^{-il\varphi}$$
$$\psi_k'' = -\xi_k^{-1}\sum_1^{\infty} l\psi_{l,k}\eta_k^{-l}, \quad \psi_{l,k} = \text{const}$$

In the first (second) boundary-value problem, Eqs (4.6) ((4.7)) reduce to a linear system of equations of order 8 for the coefficients of $e^{\pm il\varphi}$, with unknowns Re $\psi_{l,k}$; Im $\psi_{l,k}$; $k = 1, \ldots, 4$. The matrix of the system is independent of l and is non-singular by virtue of Theorem 6. When $l \geq 2$ the coefficients are uniquely defined and it follows from the absolute and uniform convergence of the Fourier series for the boundary conditions that the series (7.7) also converge absolutely and uniformly [10]. When $l = 1$ one needs more information about the form of the functions $\upsilon_\alpha^0(\mathbf{x})$. It follows from the condition $\varepsilon_{\alpha\beta}, \theta_{\alpha\beta} \to 0, |\mathbf{x}| \to \infty$ that $\upsilon_\alpha^0 = \pm\, \omega_{21}x_\beta$, $\omega_{21} = \text{const} \in R$; $Q^0_{\alpha\beta}, M^0_{\alpha\beta} = 0$.

First boundary-value problem. The eight equations (4.6) contain nine unknowns Re $\psi_{1,k}$, Im $\psi_{1,k}$ and ω_{21}. Of the two additional single-valuedness equations (5.1), the first is not independent by condition (7.2) an drops out. The remaining system of equations yields two unknowns. When $l = 0$ we obtain just four equations (4.6) and the coefficients ψ_k remain undetermined. However, physical meaning can only be attached to the linear combinations of these coefficients corresponding to translational shear in the longitudinal plane and rotation of the plate as a rigid body about the x_1, x_2 axes. These components are determined from the equalities

$$2\operatorname{Re}\left\{\Sigma\frac{p_\alpha}{p_0}(s_k)\psi_{0,k}\right\} = u_{0,\alpha};\quad w_0 + 2\operatorname{Re}\{\Sigma\zeta_k\psi_{0,k}\} = w_0 + x_\alpha w_{0,\alpha}$$

$$w_t = w_0 - \int\limits_{(0^*,t)} \theta^*_\alpha dx_\alpha \tag{7.8}$$

Translational shear along the vertical is found from the given values of the deflection at the points of the contour $(0^*, t) \subset \partial\Omega$.

Second boundary-value problem. When $l = 1$ we obtain eight equations (4.7) and two equations (5.1), of which the second is dependent because of (7.4), $\mathcal{M}_3 = 0$ and the single-valuedness of the longitudinal stresses. The final result comprises the unknowns Re $\psi_{1,k}$, Im $\psi_{1,k}$; $c \in R$. The quantities $\psi_{0,k}$, ω_{21} and w_0 remain undetermined.

8. Non-self-balanced loads may be dealt with using formulae (6.1). To avoid the need to correct the previous arguments, it is more convenient to replace the logarithmic function as follows:

$$\Psi_k(\zeta_k) = \tfrac{1}{4}A_k\{(2\zeta_k^2 - e_k^2)\ln\eta_k + \xi_k\zeta_k\} + B_k\{\zeta_k\ln\eta_k - \xi_k\} \tag{8.1}$$

$$\Delta_k''' = 0,\quad \Delta_k'' = 2\pi i A_k,\quad \Delta_k' = 2\pi i B_k + \zeta_k\Delta_k''$$

$$\Delta_k = \pi i(a_1 - i\lambda_k a_2)\psi_{1,k} + \zeta_k\Delta_k' + \tfrac{1}{4}(2\zeta_k^2 - e_k^2)\Delta_k''$$

The coefficients A_k and B_k are determined in similar fashion.

9. In the limit of an infinitely narrow ellipse we obtain solutions of boundary-value problems for a finite cutout $-a_1 \leqslant x_1 \leqslant a_1, x_2 = 0$ in an infinite plate. The previous systems of equations are considerably simplified. We proceed now to an "averaged" analogue of the singular problem, which may be understood as an asymptotic limit of the internal SSS as $a_2 \to +0$, $h/a_2 \to +0$. One corollary of formulae (4.2) and (7.7) is that the stress resultants and moments $Q_{\alpha\beta}$ and $M_{\alpha\beta}$ (and the stress components $\sigma^j_{\alpha\beta}$) at the tip of the cutout have a typical singularity of order $r^{-1/2}$, where r is the distance to the final point

$$x_1 = a_1\cos\varphi + r\cos\mu,\quad x_2 = a_2\sin\varphi + r\sin\mu,\quad 0 \leqslant \mu \leqslant 2\pi$$

$$\xi_k = \{\pm 2a_1 r(\cos\mu - \lambda_k\sin\mu) + O(a_2^2) + O(r^2)\}^{1/2},\quad \varphi = 0, \pi$$

$$\psi_k''(\zeta_k) \sim \xi_k^{-1} = O(r^{-1/2}),\quad Q_{\alpha\beta} \sim r^{-1/2} f_{\alpha\beta}(\mu)$$

At other points the functions are regular. Averaging of the quantities $\sigma^j_{\alpha z}$, σ^j_{zz} and stress resultants $Q_{\alpha z}$ considerably distorts their asymptotic behaviour as $r \to 0$.

The treatment for a rigid inclusion is entirely analogous.

10. The interior of an ellipse with boundary $\partial\Omega$ and cut $(-e_k, e_k)$ may be transformed into an annulus by a mapping $\eta_k(\zeta)_k)$: $|b_k| \leqslant |\eta_k| \leqslant 1$ [11]. The unknown functions are defined by a Laurent series, assuming that $\mathcal{M} = 0$, $\mathcal{F} = 0$

$$\psi_k(\zeta_k) = \text{const} + \psi_{0,k}\zeta_k + \tfrac{1}{2}\{\psi_{1,k}\zeta_k^2 + (a_1 - i\lambda_k a_2)\times$$

$$\times\sum_2^\infty \psi_{l,k}\left[\frac{\eta_k^{l+1} + (b_k\eta_k^{-1})^{l+1}}{l+1} - b_k\frac{\eta_k^{l-1} + (b_k\eta_k^{-1})^{l-1}}{l-1}\right]\Bigg\}$$

$$\psi'_k = \psi_{0,k} + \psi_{1,k}\zeta_k + \sum_2^\infty \psi_{l,k}[\eta_k^l + (b_k\eta_k^{-1})^l]$$

$$\psi''_k = \psi_{1,k} + \xi_k^{-1}\sum_2^\infty l\psi_{l,k}[\eta_k^l - (b_k\eta_k^{-1})^l]$$

It is obvious that the structure of the potentials $\psi'_k(\zeta_k)$ coincides exactly with the solution of S. G. Lekhnitskii (expansion in Faber polynomials) and P. P. Kufarev for a single-ply elliptical plate.

We again obtain eight linear equations (4.6) or (4.7) for the coefficients Re $\varphi_{l,k}$ and Im $\psi_{l,k}$. When $l \geqslant 2$ the required quantities are uniquely defined; the fact that the system is non-singular follows from Theorem 6.

Note that in the limit of $(l \to \infty, b_k^l \to 0)$ the matrix of the system of equations is identical with the matrix for the problem of a plate with a cutout. Thus it follows from the absolute and uniform convergence of the Fourier series for the boundary conditions that the Laurent series are also absolutely and uniformly convergent.

Another observation is that the cutout $(-e_k, e_k)$ was made for formal convenience, but the final expressions do not involve any odd-ordered radicals and the points of the cutout are identified. Since the single-valued functions $\psi_k(\zeta_k)$ contain quadratic trinomial, we can set $v_\alpha^0(\mathbf{x}) = 0$, $Q_{\alpha\beta}^0 = 0$, $Q_{\alpha\beta}^0$, $M_{\alpha\beta}^0 = 0$. This is a general property of the representations (4.2); it is easy to show that, due to the quadratic components, there is an adequate degree of arbitrariness and the omitted terms become superfluous.

First boundary-value problem. If $l = 1$, one of the eight equations (4.6) is the first condition of (7.2). If $l = 0$, only four equations (4.6) remain. One cannot determine the coefficients $\psi_{1,k}$ themselves, but only some of their linear combinations these, together with condition (7.8), yield the linear components of the longitudinal displacements and quadratic components of the deflection

$$u_\alpha = (u_{1,\alpha} + u_{-1,\alpha})x_1 a_1^{-1} + i(u_{1,\alpha} - u_{-1,\alpha})x_2 a_2^{-1} + u_{0,\alpha}$$

$$2w = i(w_{1,\alpha} - w_{-1,\alpha})x_\alpha^2 a_\alpha^{-1} + [i(w_{1,1} - w_{-1,1})a_2^{-1} + (w_{1,2} + w_{-1,2})a_1^{-1}]x_1 x_2 + 2(x_\alpha w_{0,\alpha} + w_0)$$

Second boundary-value problem. The quantities Re $\psi_{1,k}$, Im $\psi_{1,k}$ and c $(l = 1)$ are to be determined from eight equations (4.7), but one of the latter is dependent because of condition (7.4) and the fact that $\mathcal{M}_3 = 0$. The required constants remain undetermined, but one can find all the appropriate (constant) components of the stress resultants and moments $Q_{\alpha\beta}$ and $M_{\alpha\beta}$; the strain and curvature components are determined from (1.4). The other coefficients and the components of the translational shears and rotation of the plate as a rigid whole are obviously unknown.

11. To end this paper we will formulate a general result that yields exact solutions for domains that can be mapped conformally onto a circle. Denote the coefficients of the functions $\psi'_k(\zeta_k)$ in Eqs (4.6) or (4.7) by a_{lk}, $B = \| a_{lk} \|^{-1}$ (singularity of the matrix would imply that the solution of the previous problem—a plate with a cutout—was not unique). Let H_l^0 $(l = 1, \ldots, 4)$ be the right-hand sides of Eqs (4.6) or (4.7).

Theorem 7. Suppose that the following conditions hold:

1. a set of conformal mappings of the domains Ω_k onto the unit circle K_1 exists;
2. the external loads are self-balanced and the functions $\varphi'_k(\zeta_k)$ are single-valued;
3. the boundary conditions are stated in terms of infinitely smooth functions.

Then the unknown functions are

$$\psi'_k(\zeta_k) = \frac{1}{4\pi}\int_0^{2\pi}\sum_{l=1}^4 b_{kl}H_l^*(e^{i\varphi})\frac{e^{i\varphi} + \eta_k(\zeta_k)}{e^{i\varphi} - \eta_k(\zeta_k)}d\varphi + i\sum_{l=1}^4 b_{kl}r_l$$

$$\eta_k : \Omega_k \to K_1 = \{\rho e^{i\varphi},\ \rho \geqslant 1\},\quad \kappa_k : K_1 \to \Omega_k,\quad H_l^0(x)|_{\partial\Omega} = H_l^0(\kappa_k(e^{i\varphi})) = H_l^*$$

where $\mathbf{r} = \{r_l\}$ is a vector of arbitrary real constants.

The proof follows from Schwartz's formula (the determination of an analytic function in a circle given its real part on the circumference). By Riemann's theorem, the aforementioned conformal mappings always exist if the initial domain Ω is simply connected and has a smooth boundary [14].

The question of the effective construction of such mappings for any domain Ω_k remains an open one; this cannot be done even for a finite ellipse, and use is made of Faber polynomials.

12. The method proposed above is no different in its underlying idea from the Kolosov–Muskhelishvili–Lekhnitskii method of classical complex potentials. However, the potential is defined directly for the displacements (without an analogue of the Airy function), and the problem of coupled bending–extension–compression-shear of a plate is of total dimension unlike the separate classical consideration. The determination of the constants of integration is somewhat different. In all other respects the methods are analogous, down to preservation of the structure of the potentials in the same form as in the corresponding two-dimensional or bending problems. One can thus make effective use of the available stock of solutions.

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